Bessel Differential Equation: General Solution and Derivation of the Neumann Function for Integer Orders

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Abstract

We derive the general solution of the Bessel equation for all orders via the Frobenius method. In particular, for integer orders, we obtain the Bessel function of the second kind (Neumann function), which serves as the second linearly independent solution alongside the Bessel function of the first kind. Finally, it is concluded that the universal general solution is given by the linear combination of both functions.

1 Description of the Bessel Equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0. \tag{1}$$

The Bessel equation is a second-order linear homogeneous ODE with variable coefficients. In order to solve this type of equation, it is natural to propose a series solution. The solutions are related to the parameter ν ($\nu \geq 0$), which determines the order of the Bessel equation. Our objective is to derive the general solution of the Bessel equation $\forall\nu$.

2 Series Approach

In order to construct this series, it is convenient for typical applied problems to center it at $x = 0$ in the derivation of the general solution. However, we must first determine the nature of the point $x = 0$. If $x = 0$ is an ordinary point, then a power series solution is valid. Otherwise, it is classified as a singular point and we must proceed with a different approach. To determine whether $x = 0$ is an ordinary point, it is convenient to express Eq. (1) in the form

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{x^2-\nu^2}{x^2}y=0. \tag{2}$$

We evaluate the limits of the coefficients of $\frac{dy}{dx}$ and y as $x \to 0$. If both limits exist, then $x = 0$ is an ordinary point.

$$\lim_{x\to 0}\frac{1}{x}=\pm\infty,$$

$$\lim_{x\to 0}\frac{x^2-\nu^2}{x^2}=\begin{cases}1 & \text{if} \quad \nu=0\\ -\infty & \text{if} \quad \nu>0.\end{cases}$$

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Thus, $x = 0$ is a singular point, implying that a standard power series is not valid as a complete solution for the equation.

Nonetheless, there are two kinds of singular points: regular and irregular. If $x = 0$ is a regular singular point, the limits as $x \to 0$ of the coefficients of $\frac{dy}{dx}$ and y in Eq. (2), multiplied by x and x^2 respectively, exist.

$$\lim_{x\to 0} x\frac{1}{x} = \lim_{x\to 0} 1 = 1,$$

$$\lim_{x\to 0} x^2\,\frac{x^2-\nu^2}{x^2} = \lim_{x\to 0}(x^2-\nu^2) = -\nu^2.$$

Hence, $x = 0$ is a regular singular point. Therefore, according to Fuch's theorem, it is valid to apply the Frobenius method to obtain the general solution of the Bessel equation.

3 The Frobenius Method

Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r},$$

where r is a real number to be determined.

We calculate the first and second derivatives to substitute into Eq. (1).

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + (x^2-\nu^2)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0. \tag{3}$$

In order to find a relation between the coefficients of the series, all terms must have the same power of x. It is convenient to choose x^{n+r}, as it appears in most of the terms. Thus, we shift the index of the series

$$\sum_{n=0}^{\infty} a_n x^{n+r+2}$$

by substituting $n \to n-2$, which yields

$$\sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{n=2}^{\infty} a_{n-2}\, x^{n+r}.$$

Inserting this result into Eq. (3) gives

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}\, x^{n+r} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

We separate all the terms corresponding to $n = 0$ and $n = 1$, combining the remainder into a single series.

$$a_0[r(r-1) + r - \nu^2]x^r + a_1[r(r+1) + r + 1 - \nu^2]x^{r+1}$$
$$+ \sum_{n=2}^{\infty}[a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \nu^2 a_n]x^{n+r} = 0.$$

Due to linear independence of $\{x^{n+r}\}$, the coefficients of each term must be zero.

$$a_0[r(r-1) + r - \nu^2] = 0, \tag{4}$$

$$a_1[r(r+1) + r + 1 - \nu^2] = 0, \tag{5}$$

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \nu^2 a_n = 0, \quad n = 2, 3, 4, \ldots. \tag{6}$$

From Eq. (4), which corresponds to $n = 0$, we find the indicial equation:

$$r(r-1) + r - \nu^2 = 0, \tag{7}$$

where we have divided both sides by a_0, since $a_0 \neq 0$ in the Frobenius method.

By simplifying Eq. (7), we obtain the values of r.

$$r^2 - \nu^2 = 0 \tag{8}$$

$$r = \pm\nu. \tag{9}$$

From Eq. (5) we have

$$a_1[r(r+1) + r + 1 - \nu^2] = 0$$
$$a_1[r^2 + 2r + 1 - \nu^2] = 0.$$

We reorder terms in a convenient form.

$$a_1[(r^2 - \nu^2) + 2r + 1] = 0.$$

Employing Eq. (8), we reduce the equation, resulting in

$$a_1(2r + 1) = 0$$

$$a_1 = 0 \quad \text{or} \quad r = -\frac{1}{2}.$$

Therefore, $a_1 = 0$ for all orders ν, except for the case where $\nu = \frac{1}{2}$. However, for $r = -\frac{1}{2}$, a_1 is not necessarily zero, and its value is arbitrary.

3.1 The Recurrence Relation and the Series Coefficients

Now, we will work with Eq. (6) in a general manner, keeping r without substituting the values of Eq. (9), until case-by-case analysis is required.

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \nu^2 a_n = 0, \quad n = 2, 3, 4, \ldots.$$

We factor out a_n, yielding

$$a_n\left[(n+r)(n+r-1) + n + r - \nu^2\right] + a_{n-2} = 0$$

$$a_n\left\{(n+r)[(n+r-1) + 1] - \nu^2\right\} + a_{n-2} = 0$$

$$a_n\left[(n+r)^2 - \nu^2\right] + a_{n-2} = 0$$

$$a_n\left[n^2 + 2nr + \underbrace{r^2 - \nu^2}_{0}\right] + a_{n-2} = 0.$$

Therefore, we obtain the recurrence relation:

$$a_n = -\frac{a_{n-2}}{n(n+2r)}, \quad n = 2, 3, 4, \ldots . \tag{10}$$

The recurrence relation allows us to determine the coefficients of the series.

It is convenient to express Eq. (10) in a form that relates the n and $n+2$ terms, so we can evaluate the coefficients more easily. Still, Eq. (10) will be helpful in an analysis that we must do when the Frobenius method presents problems.

Let $n = l + 2$, where l is an integer. Then, we have

$$a_{l+2} = -\frac{a_l}{(l+2)(l+2+2r)}, \quad l + 2 = 2, 3, 4, \ldots .$$

Since l is a dummy index, we can change the index back to n,

$$a_{n+2} = -\frac{a_n}{(n+2)(2r+n+2)}, \quad n = 0, 1, 2, \ldots . \tag{11}$$

Evaluating the recurrence relation of Eq. (11) for successive values of n yields:

$$\begin{aligned}
n = 0 \quad &\Rightarrow \quad a_2 = -\frac{a_0}{4(r+1)},\\
n = 1 \quad &\Rightarrow \quad a_3 = -\frac{a_1}{3(2r+3)},\\
n = 2 \quad &\Rightarrow \quad a_4 = -\frac{a_2}{8(r+2)} = \frac{a_0}{4 \cdot 8(r+1)(r+2)},\\
n = 3 \quad &\Rightarrow \quad a_5 = -\frac{a_3}{5(2r+5)} = \frac{a_1}{3 \cdot 5(2r+3)(2r+5)},\\
n = 4 \quad &\Rightarrow \quad a_6 = -\frac{a_4}{12(r+3)} = -\frac{a_0}{4 \cdot 8 \cdot 12(r+1)(r+2)(r+3)},\\
n = 5 \quad &\Rightarrow \quad a_7 = -\frac{a_5}{7(2r+7)} = -\frac{a_1}{3 \cdot 5 \cdot 7(2r+3)(2r+5)(2r+7)}.
\end{aligned}$$

This process may continue indefinitely, obtaining more and more terms in the series. Furthermore, note that we write all the coefficients in terms of a_0 and a_1.

From the values obtained, we can identify a pattern for the n-th even and odd coefficients. To observe the pattern for even n, we express a_6 in a different form.

$$\begin{aligned}
a_6 &= -\frac{a_0}{4 \cdot 8 \cdot 12(r+1)(r+2)(r+3)}\\
&= -\frac{a_0}{2^6(1 \cdot 2 \cdot 3)(r+1)(r+2)(r+3)}\\
&= -\frac{a_0}{2^6\, 3!(r+1)(r+2)(r+3)} .
\end{aligned}$$

Therefore, for $n = 2k$, with $k = 0, 1, 2, \ldots$, we find

$$a_{2k} = (-1)^k \frac{a_0}{2^{2k}\, k!(r+1)(r+2)\cdots(r+k)}. \tag{12}$$

Similarly, from the coefficient a_7 we obtain an expression for a_{2k+1}.

$$a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!!\,(2r+3)(2r+5)\cdots[2(r+k)+1]}, \quad k = 0, 1, 2, \ldots, \tag{13}$$

where !! denotes the double factorial.

Strictly speaking, Eqs. (12) and (13) should be proven via mathematical induction, but we assume these results as valid to avoid deviating from our main objective.

A convenient expression for the Even Coefficients. As can be noticed, the coefficients a_{2k} and a_{2k+1} are written in a cumbersome form. It is not possible to directly evaluate the corresponding k to determine the value of the coefficient; this is because these expressions were deduced from an observed pattern.

The coefficient of interest is the even one, a fact that will be clarified when we analyze the different cases for the solution. To facilitate its manipulation, we express a_{2k} employing the Gamma function.

We recall the recursive property of the Gamma function:

$$\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{C}. \tag{14}$$

Taking z as $r+k$, and applying the property of Eq. (14) successively, yields

$$\begin{aligned}\Gamma(r+k+1) &= (r+k)\Gamma(r+k)\\ &= (r+k)(r+k-1)\Gamma(r+k-1)\\ &\dots\\ &= (r+k)(r+k-1)\cdots(r+2)\Gamma(r+2)\\ &= (r+k)(r+k-1)\cdots(r+2)(r+1)\Gamma(r+1)\end{aligned}$$

$$\therefore \quad (r+1)(r+2)\cdots(r+k) = \frac{\Gamma(r+k+1)}{\Gamma(r+1)}.$$

Substituting this result into Eq. (12) gives

$$a_{2k} = (-1)^k \frac{a_0\,\Gamma(r+1)}{2^{2k}\,k!\,\Gamma(r+k+1)}, \quad k = 0, 1, 2, \dots. \tag{15}$$

4 Solution for the Different Cases

We categorize the derivation into three main cases.

4.1 Case I: $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

In the Frobenius method, we anticipate complications with the series solution if the difference between the roots of the indicial equation is an integer.

We recall the solutions of the indicial equation, which are in Eq. (9).

$$r = \pm\nu.$$

Then, we have

$$\begin{aligned} r_1 = \nu, &\quad r_2 = -\nu\\ r_1 - r_2 &= 2\nu. \end{aligned} \tag{16}$$

If we substitute a half-integer value of ν into Eq. (16), we obtain

$$r_1 - r_2 = 2\nu \in \mathbb{Z}$$

Due to this, the recurrence relation presents a problem. To be precise, a division by zero occurs when we substitute the smaller root.

We recall Eq. (10).

$$a_n = -\frac{a_{n-2}}{n(n+2r)}, \quad n = 2, 3, 4, \dots.$$

Substituting $r = -\nu$ yields

$$a_n = -\frac{a_{n-2}}{n(n-2\nu)},$$

and for the term $n = 2\nu$, we find that

$$a_{2\nu} = -\frac{a_{2\nu-2}}{0}. \tag{17}$$

Since $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, the terms for which we have this issue are the odd terms ($n = 2\nu$ is an odd number). Because of the nature of the recurrence relation, the even terms remain unaffected, given the indices jump by two at each step. That is the reason we have different expressions for a_{2k} and a_{2k+1}.

In principle, the particular case $\nu = \frac{1}{2}$ is different from the other half-integer values, since the problematic term in this case is $n = 2\nu = 1$, and a_1 is the coefficient from which we calculate all the odd ones (the odd coefficients are multiples of a_1); then a_1 is not produced by the recurrence relation, so its value is arbitrary. Furthermore, this is consistent with the results obtained from Eq. (5), where we determined that a_1 must be zero $\forall\nu$, except for $\nu = \frac{1}{2}$.

In the analysis of the $\nu = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$ orders, we recall that all odd terms are multiples of a_1, according to Eq. (13), and we already know that $a_1 = 0$, except for the previous case; then, it would be natural to conclude that $a_3 = a_5 = a_7 = \cdots = 0$. However, we must pay special attention, because this is not necessarily true for all the cases.

Eq. (17) shows that there is a division by zero when we substitute into it $r = -\nu$ and $n = 2\nu$, so if all the coefficients are multiples of a_1, and $a_1 = 0$, we encounter a $\frac{0}{0}$ ratio in the $a_{2\nu}$ term when $r = -\nu$. That $\frac{0}{0}$ means that these particular coefficients are indeterminate; hence their value is arbitrary, and the resulting solution remains valid regardless of the chosen value. We can observe this more clearly in the following example.

We take the recurrence relation from Eq. (10), expressed in a valid form when there is a division by zero, and substitute $r = -\frac{3}{2}$ and $n = 2\left(\frac{3}{2}\right) = 3$.

$$a_n = -\frac{a_{n-2}}{n(n+2r)} \quad \Rightarrow \quad n(n+2r)a_n = -a_{n-2}$$

$$0 \cdot a_3 = -a_1,$$

and given that $a_1 = 0$, we obtain

$$0 \cdot a_3 = 0.$$

The above equation is always true, regardless of the value of a_3. Therefore, although we have a division by zero, this does not represent a real problem for finding the solution; it is valid to employ the standard Frobenius solution, which is:

$$y(x) = \sum_{n=0}^{\infty} a_{n_{r_1}}\, x^{n+r_1} + \sum_{n=0}^{\infty} a_{n_{r_2}}\, x^{n+r_2}, \tag{18}$$

where $a_{n_{r_1}}$ and $a_{n_{r_2}}$ represent the coefficients evaluated at their respective roots, i.e., r_1 and r_2.

This analysis explains why, in the standard literature regarding the Bessel equation, the odd coefficients are zero. It is not a mathematical necessity, but rather a choice of convenience; the simplest solution is obtained by setting $a_1 = a_3 = a_5 = \cdots = 0$.

We briefly demonstrate this fact by taking the series with $r_2 = -\nu$, specifically the term 2ν.

$$x^{r_2+2\nu} = x^{-\nu+2\nu} = x^{\nu}.$$

This term overlaps with a term in the series with $r_1 = \nu$, thus in order to avoid the overlap, it is convenient to set all the odd coefficients to zero.

Consequently, since $a_{2k+1} = 0$, Eq. (18) becomes

$$y(x) = \sum_{k=0}^{\infty} a_{2k_{r_1}}\, x^{2k+r_1} + \sum_{k=0}^{\infty} a_{2k_{r_2}}\, x^{2k+r_2}.$$

Substituting $r_1 = \nu$ and $r_2 = -\nu$ yields

$$y(x) = \sum_{k=0}^{\infty} a_{2k_\nu} x^{2k+\nu} + \sum_{k=0}^{\infty} a_{2k_{-\nu}} x^{2k-\nu},$$

where, according to Eq. (15), a_{2k_ν} and $a_{2k_{-\nu}}$ are given by:

$$a_{2k_\nu} = (-1)^k \frac{a_0\Gamma(\nu+1)}{2^{2k}\, k!\,\Gamma(\nu+k+1)}, \quad a_{2k_{-\nu}} = (-1)^k \frac{a_0\Gamma(-\nu+1)}{2^{2k}\, k!\,\Gamma(-\nu+k+1)}.$$

Then, we have

$$y(x) = \sum_{k=0}^{\infty}(-1)^k \frac{a_0\Gamma(\nu+1)}{2^{2k}\, k!\,\Gamma(\nu+k+1)} x^{2k+\nu} + \sum_{k=0}^{\infty}(-1)^k \frac{a_0\Gamma(-\nu+1)}{2^{2k}\, k!\,\Gamma(-\nu+k+1)} x^{2k-\nu}. \tag{19}$$

We invoke the definitions of the Bessel functions of the first kind for orders ν and $-\nu$:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}, \tag{20}$$

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu}. \tag{21}$$

We can identify in Eq. (19) that both series are similar to the definitions of $J_\nu(x)$ and $J_{-\nu}(x)$. In order to put this result in terms of the Bessel functions of the first kind, we do a simple algebraic manipulation.

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty}(-1)^k \frac{a_0\Gamma(\nu+1)2^\nu}{2^{2k+\nu}\, k!\,\Gamma(\nu+k+1)} x^{2k+\nu} + \sum_{k=0}^{\infty}(-1)^k \frac{a_0\Gamma(-\nu+1)2^{-\nu}}{2^{2k-\nu}\, k!\,\Gamma(-\nu+k+1)} x^{2k-\nu} \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{a_0\Gamma(\nu+1)2^\nu}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu} + \sum_{k=0}^{\infty}(-1)^k \frac{a_0\Gamma(-\nu+1)2^{-\nu}}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu}. \end{aligned}$$

Now, we define two arbitrary constants, c_1 and c_2, in the following way:

$$c_1 \equiv a_0\Gamma(\nu+1)2^\nu, \quad c_2 \equiv a_0\Gamma(-\nu+1)2^{-\nu}.$$

Substituting these values into the corresponding series yields

$$y(x) = c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu} + c_2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu},$$

Therefore, we obtain the solution as follows:

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \quad \nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots. \tag{22}$$

4.2 Case II: $\nu \neq 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

Recalling that the Frobenius method presents complications when the difference between the roots of the indicial equation is an integer, it is straightforward to notice that this is the simplest case of the general solution.

We return to Eq. (16).

$$r_1 - r_2 = 2\nu,$$

where 2ν is an integer if and only if ν is an integer or a half-integer. Then, for this case, that difference is never an integer. Hence, we can apply directly the standard solution given by the Frobenius method, i.e., Eq. (18):

$$y(x) = \sum_{n=0}^{\infty} a_{n_{r_1}} x^{n+r_1} + \sum_{n=0}^{\infty} a_{n_{r_2}} x^{n+r_2}.$$

All the subsequent procedure is the same as in Case I. Therefore, for this case, the solution is

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \quad \nu \neq 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots. \tag{23}$$

If we combine Eqs. (22) and (23), we obtain:

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \quad \nu \neq 0, 1, 2, \ldots, \tag{24}$$

which is the most general solution we have for the Bessel equation, up to this point.

4.3 Case III: $\nu = 0, 1, 2, \ldots$

This case presents a fundamental issue that manifests in two distinct ways: one arises from a property of the Bessel functions, and the other emerges from the recurrence relation.

From the theory of Bessel functions:

$$J_{-p}(x) = (-1)^p J_p(x), \quad p = 0, 1, 2, \ldots, \tag{25}$$

which means that for this case, $J_\nu(x)$ and $J_{-\nu}(x)$ are multiples of each other, i.e., they are linearly dependent. Thus, Eq. (24) is not valid as a general solution.

Furthermore, from the recurrence relation, we can observe a breakdown. Recalling Eq. (16), we have

$$r_1 - r_2 = 2\nu,$$

and for this case, this difference is an integer. Then, as in Case I, there exists a division by zero in the term $n = 2\nu$ when we substitute the smaller root.

We recall Eq. (10).

$$a_n = -\frac{a_{n-2}}{n(n+2r)}, \quad n = 2, 3, 4, \ldots.$$

Substituting $r = -\nu$ and $n = 2\nu$ yields

$$a_{2\nu} = -\frac{a_{2\nu-2}}{0},$$

with a crucial distinction: the value of $a_{2\nu-2} \neq 0$, because 2ν is even, and we already determined that the even coefficients are not zero. Hence, this situation is more complex than in Case I, since dividing a non-zero number by zero results in something that does not exist. We can see this more clearly writing the recurrence relation in the following form:

$$n(n+2r)a_n = -a_{n-2},$$

and substituting $r = -\nu$ and $n = 2\nu$ gives

$$0 \cdot a_{2\nu} = -a_{2\nu-2}.$$

Since $a_{2\nu-2} \neq 0$, there is no value for $a_{2\nu}$ that satisfies the above equation. Therefore, unlike Case I, where the $\frac{0}{0}$ implies arbitrary coefficients, here, the solution of Eq. (18) breaks down.

According to the Frobenius method, there is a procedure to obtain a second linearly independent solution for this case, thereby solving the issue of the linear dependence between $J_\nu(x)$ and $J_{-\nu}(x)$. That procedure consists in proposing a second solution of the form:

$$y_2(x) = y_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \tag{26}$$

where b_n are arbitrary constants that we must determine by substituting $y_2(x)$ into the Bessel equation. Additionally, we explicitly write r_2 in the series, because it is the smaller root; consequently, $y_1(x) = J_\nu(x)$ and $y_2(x)$ is a solution constructed to be linearly independent from $J_\nu(x)$, unlike $J_{-\nu}(x)$.

However, the approach we will follow is more standard in Bessel theory. In practice, the second linearly independent solution is not typically expressed in the form of Eq. (26), but rather through the Neumann function, $Y_\nu(x)$.

The Neumann function, also known as the Bessel function of the second kind, was introduced with the objective of solving the linear dependence between $J_\nu(x)$ and $J_{-\nu}(x)$ for integer ν. It is defined as follows:

$$Y_\nu(x) = \frac{\cos(\nu\pi)\, J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{27}$$

This particular form of the definition is motivated by what happens to Eq. (27) when ν is an integer. We briefly demonstrate this fact below.

Let $\nu = p$, where $p = 0, 1, 2, \dots$. Then, by substituting into Eq. (27), we have

$$Y_p(x) = \frac{\cos(p\pi)\, J_p(x) - J_{-p}(x)}{\sin(p\pi)},$$

and employing the identity $\cos(p\pi) = (-1)^p$ yields

$$Y_p(x) = \frac{(-1)^p J_p(x) - J_{-p}(x)}{\sin(p\pi)}.$$

We recall Eq. (25), resulting in

$$Y_p(x) = \frac{J_{-p}(x) - J_{-p}(x)}{\sin(p\pi)}.$$

Due to $\sin(p\pi) = 0$, we obtain

$$Y_p(x) = \frac{0}{0}.$$

By direct evaluation, we get an indeterminate form, unlike previously, where we obtained an absurdity. We exploit this freedom by taking the limit as $\nu \to p$. Therefore, we must study the behavior of the Neumann function in the vicinity of integer ν.

$$Y_p(x) = \lim_{\nu \to p} \frac{\cos(\nu\pi)\, J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}, \quad p = 0, 1, 2, \dots. \tag{28}$$

Since we already showed that direct evaluation yields a $\frac{0}{0}$ indeterminacy, it is valid to apply L'Hôpital's rule to the limit.

$$\begin{aligned} Y_p(x) &= \lim_{\nu \to p} \frac{\frac{\partial}{\partial \nu}\left[\cos(\nu\pi)\, J_\nu(x) - J_{-\nu}(x)\right]}{\frac{\partial}{\partial \nu}\sin(\nu\pi)} \\ &= \lim_{\nu \to p} \frac{-\pi \sin(\nu\pi)\, J_\nu(x) + \cos(\nu\pi)\, \frac{\partial}{\partial \nu} J_\nu(x) - \frac{\partial}{\partial \nu} J_{-\nu}(x)}{\pi \cos(\nu\pi)}. \end{aligned}$$

Given

$$\lim_{\nu \to p} \pi \cos(\nu\pi) \neq 0,$$

we can apply properties of limits to evaluate it and simplify the expression.

$$Y_p(x) = \frac{1}{\pi \cos(p\pi)} \lim_{\nu \to p} \left[-\pi \sin(\nu\pi)\, J_\nu(x) + \cos(\nu\pi)\, \frac{\partial}{\partial \nu} J_\nu(x) - \frac{\partial}{\partial \nu} J_{-\nu}(x)\right],$$

and because

$$\lim_{\nu \to p} \sin(\nu\pi) = 0, \quad \cos(p\pi) = (-1)^p,$$

we find

$$Y_p(x) = \frac{(-1)^p}{\pi} \lim_{\nu \to p} \left[\cos(\nu\pi)\, \frac{\partial}{\partial \nu} J_\nu(x) - \frac{\partial}{\partial \nu} J_{-\nu}(x)\right]. \tag{29}$$

For clarity, we calculate the derivatives $\frac{\partial}{\partial \nu} J_\nu(x)$ and $\frac{\partial}{\partial \nu} J_{-\nu}(x)$ separately.

Calculation of $\frac{\partial}{\partial\nu}J_\nu(x)$ We recall the definition of $J_\nu(x)$ from Eq. (20),

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}.$$

Taking the derivative with respect to ν yields

$$\begin{aligned}\frac{\partial}{\partial\nu}J_\nu(x) &= \frac{\partial}{\partial\nu}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\partial}{\partial\nu}\left[\Gamma(\nu+k+1)^{-1}\left(\frac{x}{2}\right)^{2k+\nu}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\nu}J_\nu(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Bigg\{&-\Gamma(\nu+k+1)^{-2}\left(\frac{x}{2}\right)^{2k+\nu}\left[\frac{\partial}{\partial(\nu+k+1)}\Gamma(\nu+k+1)\right]\underbrace{\frac{\partial}{\partial\nu}(\nu+k+1)}_{1}\\ &+\Gamma(\nu+k+1)^{-1}\frac{\partial}{\partial\nu}\left(\frac{x}{2}\right)^{2k+\nu}\Bigg\},\end{aligned} \tag{30}$$

where we have employed the derivative of a product and the chain rule.

We simplify the expression by using the derivative of a general exponential function:

$$\frac{d}{du}a^u = a^u\ln(a),$$

which, for our case, yields

$$\frac{\partial}{\partial\nu}\left(\frac{x}{2}\right)^{2k+\nu} = \left(\frac{x}{2}\right)^{2k+\nu}\ln\left(\frac{x}{2}\right).$$

Substituting this result into Eq. (30) gives

$$\begin{aligned}\frac{\partial}{\partial\nu}J_\nu(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Bigg\{&-\Gamma(\nu+k+1)^{-2}\left(\frac{x}{2}\right)^{2k+\nu}\left[\frac{\partial}{\partial(\nu+k+1)}\Gamma(\nu+k+1)\right]\\ &+\Gamma(\nu+k+1)^{-1}\left(\frac{x}{2}\right)^{2k+\nu}\ln\left(\frac{x}{2}\right)\Bigg\}.\end{aligned} \tag{31}$$

In order to work with the term

$$\frac{\partial}{\partial(\nu+k+1)}\Gamma(\nu+k+1),$$

we invoke the Digamma function, which is defined as follows:

$$\psi(z) \equiv \frac{d}{dz}\ln\Gamma(z), \quad z\in\mathbb{C}.$$

For our derivation, it is convenient to write it as

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{32}$$

We take $z = \nu+k+1$ and substitute into Eq. (31) to obtain

$$\frac{\partial}{\partial\nu}J_\nu(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left[-\Gamma(\nu+k+1)^{-1}\left(\frac{x}{2}\right)^{2k+\nu}\psi(\nu+k+1)+\Gamma(\nu+k+1)^{-1}\left(\frac{x}{2}\right)^{2k+\nu}\ln\left(\frac{x}{2}\right)\right].$$

Rearranging terms yields

$$\frac{\partial}{\partial\nu}J_\nu(x) = \underbrace{\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu}}_{J_\nu(x)}\ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty}\frac{(-1)^k\psi(\nu+k+1)}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

$$\therefore \quad \frac{\partial}{\partial\nu}J_\nu(x) = J_\nu(x)\ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty}\frac{(-1)^k\psi(\nu+k+1)}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu}. \tag{33}$$

Calculation of $\frac{\partial}{\partial\nu}J_{-\nu}(x)$. We follow a similar procedure as for the previous derivative.

We recall the definition of $J_{-\nu}(x)$ from Eq. (21),

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu},$$

and taking the derivative with respect to ν yields

$$\begin{aligned}\frac{\partial}{\partial\nu}J_{-\nu}(x) &= \frac{\partial}{\partial\nu}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\partial}{\partial\nu}\left[\Gamma(-\nu+k+1)^{-1}\left(\frac{x}{2}\right)^{2k-\nu}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\nu}J_{-\nu}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Bigg\{&-\Gamma(-\nu+k+1)^{-2}\left(\frac{x}{2}\right)^{2k-\nu}\left[\frac{\partial}{\partial(-\nu+k+1)}\Gamma(-\nu+k+1)\right]\underbrace{\frac{\partial}{\partial\nu}(-\nu+k+1)}_{-1}\\ &+\Gamma(-\nu+k+1)^{-1}\frac{\partial}{\partial\nu}\left(\frac{x}{2}\right)^{2k-\nu}\Bigg\}.\end{aligned} \tag{34}$$

Substituting

$$\frac{\partial}{\partial\nu}\left(\frac{x}{2}\right)^{2k-\nu} = -\left(\frac{x}{2}\right)^{2k-\nu}\ln\left(\frac{x}{2}\right)$$

into Eq. (34) gives

$$\begin{aligned}\frac{\partial}{\partial\nu}J_{-\nu}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Bigg\{&\Gamma(-\nu+k+1)^{-2}\left(\frac{x}{2}\right)^{2k-\nu}\left[\frac{\partial}{\partial(-\nu+k+1)}\Gamma(-\nu+k+1)\right]\\ &-\Gamma(-\nu+k+1)^{-1}\left(\frac{x}{2}\right)^{2k-\nu}\ln\left(\frac{x}{2}\right)\Bigg\}.\end{aligned} \tag{35}$$

We recall Eq. (32):

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

with $z = -\nu + k + 1$, for this calculation. We substitute into Eq. (35) to obtain

$$\frac{\partial}{\partial\nu}J_{-\nu}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left[\Gamma(-\nu+k+1)^{-1}\left(\frac{x}{2}\right)^{2k-\nu}\psi(-\nu+k+1) - \Gamma(-\nu+k+1)^{-1}\left(\frac{x}{2}\right)^{2k-\nu}\ln\left(\frac{x}{2}\right)\right].$$

Rearranging terms yields

$$\frac{\partial}{\partial\nu}J_{-\nu}(x) = -\underbrace{\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{2k-\nu}}_{J_{-\nu}(x)}\ln\left(\frac{x}{2}\right) + \sum_{k=0}^{\infty}\frac{(-1)^k\psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{2k-\nu}$$

$$\therefore\quad \frac{\partial}{\partial\nu}J_{-\nu}(x) = -J_{-\nu}(x)\,\ln\left(\frac{x}{2}\right) + \sum_{k=0}^{\infty}\frac{(-1)^k\psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{2k-\nu}. \tag{36}$$

Substitution into the Neumann Function. We recall Eq. (29):

$$Y_p(x) = \frac{(-1)^p}{\pi} \lim_{\nu\to p} \left[\cos(\nu\pi)\,\frac{\partial}{\partial\nu} J_\nu(x) - \frac{\partial}{\partial\nu} J_{-\nu}(x)\right].$$

By substituting the results of Eqs. (33) and (36), we find

$$\begin{aligned} Y_p(x) = \frac{(-1)^p}{\pi} \lim_{\nu\to p} \Bigg\{ &\cos(\nu\pi) \left[J_\nu(x) \ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \psi(\nu+k+1)}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu} \right] \\ &+ J_{-\nu}(x)\, \ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu} \Bigg\}. \end{aligned}$$

Distributing the limit into the sums yields

$$\begin{aligned} Y_p(x) = \frac{(-1)^p}{\pi} \Bigg\{ &\lim_{\nu\to p} \cos(\nu\pi) \left[J_\nu(x) \ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \psi(\nu+k+1)}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu} \right] \\ &+ \lim_{\nu\to p} J_{-\nu}(x)\, \ln\left(\frac{x}{2}\right) - \lim_{\nu\to p} \sum_{k=0}^{\infty} \frac{(-1)^k \psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu} \Bigg\}. \end{aligned} \tag{37}$$

Note that none of the terms presents a problem when we evaluate directly, except for

$$\sum_{k=0}^{\infty} \frac{(-1)^k \psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu}.$$

If we let $\nu = p$, this series exhibits an issue for certain values of k; specifically, we encounter the following indeterminate form:

$$\frac{\psi(-p+k+1)}{\Gamma(-p+k+1)} = \frac{\infty}{\infty} \quad \text{if} \quad 0 \le k \le p-1.$$

Hence, it is convenient to isolate those singularities in another sum, i.e.,

$$\sum_{k=0}^{\infty} \frac{(-1)^k \psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu} = \sum_{k=0}^{p-1} \frac{(-1)^k \psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu} + \sum_{k=p}^{\infty} \frac{(-1)^k \psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu}.$$

Therefore, substituting the above expression into Eq. (37) and evaluating the safe limits directly yields

$$\begin{aligned} Y_p(x) = \frac{(-1)^p}{\pi} \Bigg\{ &(-1)^p \left[J_p(x) \ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \psi(p+k+1)}{k!\,\Gamma(p+k+1)} \left(\frac{x}{2}\right)^{2k+p} \right] + J_{-p}(x)\, \ln\left(\frac{x}{2}\right) \\ &- \sum_{k=p}^{\infty} \frac{(-1)^k \psi(-p+k+1)}{k!\,\Gamma(-p+k+1)} \left(\frac{x}{2}\right)^{2k-p} - \lim_{\nu\to p} \sum_{k=0}^{p-1} \frac{(-1)^k \psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu} \Bigg\}. \end{aligned} \tag{38}$$

Since the term that requires special attention is

$$\lim_{\nu\to p} \sum_{k=0}^{p-1} \frac{(-1)^k \psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu},$$

we treat it separately.

$$\lim_{\nu\to p} \sum_{k=0}^{p-1} \frac{(-1)^k \psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu} = \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \lim_{\nu\to p} \left[\frac{\psi(-\nu+k+1)}{\Gamma(-\nu+k+1)}\right] \left(\frac{x}{2}\right)^{2k-p}.$$

We invoke a property, which is demonstrated in the Appendix, that aids us in evaluating this limit:

$$\lim_{z\to -m} \frac{\psi(z)}{\Gamma(z)} = (-1)^{m+1}\, m!, \tag{39}$$

where $-m$ represents the values for which the functions $\psi(z)$ and $\Gamma(z)$ present a pole, i.e., $m = 0, 1, 2, \dots$.

For our case, we have

$$z = -\nu + k + 1 \quad \text{and} \quad -m = -p + k + 1.$$

Then, when $\nu \to p$, $z \to -m$; so the limit of Eq. (39) becomes

$$\lim_{\nu\to p} \frac{\psi(-\nu + k + 1)}{\Gamma(-\nu + k + 1)} = (-1)^{p-k}\, (p - k - 1)!. \tag{40}$$

Thus, employing Eq. (40) gives

$$\lim_{\nu\to p} \sum_{k=0}^{p-1} \frac{(-1)^k \psi(-\nu + k + 1)}{k!\,\Gamma(-\nu + k + 1)} \left(\frac{x}{2}\right)^{2k-\nu} = (-1)^p \sum_{k=0}^{p-1} \frac{(p - k - 1)!}{k!} \left(\frac{x}{2}\right)^{2k-p}. \tag{41}$$

By inserting the result from Eq. (41) into Eq. (38), we obtain:

$$\begin{aligned} Y_p(x) = \frac{(-1)^p}{\pi} \Bigg\{ &(-1)^p \left[J_p(x) \ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \psi(p + k + 1)}{k!\,\Gamma(p + k + 1)} \left(\frac{x}{2}\right)^{2k+p} \right] + J_{-p}(x)\, \ln\left(\frac{x}{2}\right) \\ &- \sum_{k=p}^{\infty} \frac{(-1)^k \psi(-p + k + 1)}{k!\,\Gamma(-p + k + 1)} \left(\frac{x}{2}\right)^{2k-p} - (-1)^p \sum_{k=0}^{p-1} \frac{(p - k - 1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \Bigg\}. \end{aligned}$$

Now, we distribute $(-1)^p$ and simplify the expression.

$$\begin{aligned} Y_p(x) = \frac{1}{\pi} \Bigg[&J_p(x) \ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \psi(p + k + 1)}{k!\,\Gamma(p + k + 1)} \left(\frac{x}{2}\right)^{2k+p} + \underbrace{(-1)^p J_{-p}(x)}_{J_p(x)}\, \ln\left(\frac{x}{2}\right) \\ &- (-1)^p \sum_{k=p}^{\infty} \frac{(-1)^k \psi(-p + k + 1)}{k!\,\Gamma(-p + k + 1)} \left(\frac{x}{2}\right)^{2k-p} - \sum_{k=0}^{p-1} \frac{(p - k - 1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \Bigg] \end{aligned}$$

$$\begin{aligned} Y_p(x) = \frac{1}{\pi} \Bigg[&2\, J_p(x) \ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \psi(p + k + 1)}{k!\,\Gamma(p + k + 1)} \left(\frac{x}{2}\right)^{2k+p} \\ &- (-1)^p \sum_{k=p}^{\infty} \frac{(-1)^k \psi(-p + k + 1)}{k!\,\Gamma(-p + k + 1)} \left(\frac{x}{2}\right)^{2k-p} - \sum_{k=0}^{p-1} \frac{(p - k - 1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \Bigg]. \end{aligned} \tag{42}$$

It is possible to simplify Eq. (42) by applying a convenient change of index to

$$\sum_{k=p}^{\infty} \frac{(-1)^k \psi(-p + k + 1)}{k!\,\Gamma(-p + k + 1)} \left(\frac{x}{2}\right)^{2k-p}.$$

Let $s = -p + k$; then $k = p$ implies $s = 0$.

$$\begin{aligned} \sum_{k=p}^{\infty} \frac{(-1)^k \psi(-p + k + 1)}{k!\,\Gamma(-p + k + 1)} \left(\frac{x}{2}\right)^{2k-p} &= \sum_{s=0}^{\infty} \frac{(-1)^{s+p}\, \psi(s + 1)}{(s + p)!\,\Gamma(s + 1)} \left(\frac{x}{2}\right)^{2(s+p)-p} \\ &= (-1)^p \sum_{s=0}^{\infty} \frac{(-1)^s\, \psi(s + 1)}{(s + p)!\,\Gamma(s + 1)} \left(\frac{x}{2}\right)^{2s+p}. \end{aligned}$$

Consequently, we find

$$\sum_{k=p}^{\infty} \frac{(-1)^k \psi(-p+k+1)}{k!\,\Gamma(-p+k+1)} \left(\frac{x}{2}\right)^{2k-p} = (-1)^p \sum_{k=0}^{\infty} \frac{(-1)^k\, \psi(k+1)}{(k+p)!\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+p}.$$

Substituting this result into Eq. (42) yields

$$\begin{aligned} Y_p(x) = \frac{1}{\pi}\Bigg[& 2\,J_p(x) \ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \psi(p+k+1)}{k!\,\Gamma(p+k+1)} \left(\frac{x}{2}\right)^{2k+p} \\ & - \sum_{k=0}^{\infty} \frac{(-1)^k\, \psi(k+1)}{(k+p)!\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+p} - \sum_{k=0}^{p-1} \frac{(p-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \Bigg]. \end{aligned} \tag{43}$$

Note that we can now combine the series. For convenience, we first use the following property of the Gamma function to further simplify the expression.

$$\Gamma(l+1) = l! \quad \text{for} \quad l = 0, 1, 2, \ldots.$$

Hence, we can write Eq. (43) as

$$\begin{aligned} Y_p(x) = \frac{1}{\pi}\Bigg[& 2\,J_p(x) \ln\left(\frac{x}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \psi(p+k+1)}{k!\,(p+k)!} \left(\frac{x}{2}\right)^{2k+p} \\ & - \sum_{k=0}^{\infty} \frac{(-1)^k\, \psi(k+1)}{(k+p)!\,k!} \left(\frac{x}{2}\right)^{2k+p} - \sum_{k=0}^{p-1} \frac{(p-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \Bigg]. \end{aligned}$$

Therefore, by combining the series, we obtain:

$$\begin{aligned} Y_p(x) = \frac{1}{\pi}\Bigg\{ & 2\,J_p(x) \ln\left(\frac{x}{2}\right) - \sum_{k=0}^{p-1} \frac{(p-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \\ & - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(p+k)!} \left[\psi(p+k+1) + \psi(k+1)\right] \left(\frac{x}{2}\right)^{2k+p} \Bigg\}. \end{aligned} \tag{44}$$

Distributing the factor $\frac{1}{\pi}$ yields the standard canonical form widely found in the literature.

$$\begin{aligned} Y_p(x) = & \frac{2}{\pi}\, J_p(x) \ln\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{p-1} \frac{(p-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \\ & - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(p+k)!} \left[\psi(p+k+1) + \psi(k+1)\right] \left(\frac{x}{2}\right)^{2k+p}. \end{aligned} \tag{45}$$

Alternative Expression for $Y_p(x)$ Eq. (45) can be expressed by writing the functions $\psi(p+k+1)$ and $\psi(k+1)$ in terms of harmonic numbers. We invoke the following property of the Digamma function:

$$\psi(l+1) = -\gamma + \sum_{j=1}^{l} \frac{1}{j}, \tag{46}$$

where $\sum_{j=1}^{l} \frac{1}{j}$ is called the l-th harmonic number and is denoted as H_l. Additionally,

$$\gamma = -\psi(1) \approx 0.577215664901 \cdots$$

is known as the Euler-Mascheroni constant.

Therefore, by applying the property of Eq. (46), we have:

$$\psi(p+k+1) = -\gamma + H_{p+k}, \tag{47}$$

$$\psi(k+1) = -\gamma + H_k. \tag{48}$$

Substituting Eqs. (47) and (48) into Eq. (45) yields

$$\begin{aligned} Y_p(x) =& \frac{2}{\pi} J_p(x) \ln\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{p-1} \frac{(p-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \\ & - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(p+k)!} \left[-\gamma + H_{p+k} + -\gamma + H_k\right] \left(\frac{x}{2}\right)^{2k+p} \end{aligned}$$

$$\begin{aligned} Y_p(x) =& \frac{2}{\pi} J_p(x) \ln\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{p-1} \frac{(p-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \\ & - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(p+k)!} \left[H_{p+k} + H_k - 2\gamma\right] \left(\frac{x}{2}\right)^{2k+p}. \end{aligned}$$

Since 2γ does not depend on k, we can rearrange to obtain

$$\begin{aligned} Y_p(x) =& \frac{2}{\pi} J_p(x) \ln\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{p-1} \frac{(p-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \\ & - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(p+k)!} \left[H_{p+k} + H_k\right] \left(\frac{x}{2}\right)^{2k+p} + \frac{2\gamma}{\pi} \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(p+k)!} \left(\frac{x}{2}\right)^{2k+p}}_{J_p(x)} \end{aligned}$$

$$\begin{aligned} \therefore \quad Y_p(x) =& \frac{2}{\pi} \left[\ln\left(\frac{x}{2}\right) + \gamma\right] J_p(x) - \frac{1}{\pi} \sum_{k=0}^{p-1} \frac{(p-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} \\ & - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \left(H_{p+k} + H_k\right)}{k!\,(p+k)!} \left(\frac{x}{2}\right)^{2k+p}. \end{aligned} \tag{49}$$

Linear Independence between $J_p(x)$ and $Y_p(x)$. The logarithmic term shows that the Neumann function exhibits a fundamentally different behavior compared to the Bessel function of the first kind. It is evident that as $x \to 0$, $Y_p(x)$ diverges to $-\infty$, whereas $J_p(x)$ remains finite (see Fig. 1). Furthermore, the sum

$$\sum_{k=0}^{p-1} \frac{(p-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-p},$$

which resulted from isolating the singularities of the Gamma and Digamma functions, also contains a divergent leading term. For $p \geq 1$, the first term is proportional to x^{-p}, dominating the logarithmic contribution and resulting in a steeper descent to $-\infty$.

Therefore, we conclude that $J_p(x)$ and $Y_p(x)$ are linearly independent for $p = 0, 1, 2, \ldots$, and thus it is valid to write the solution of the Bessel equation for integer orders as

$$y(x) = \alpha_1 J_p(x) + \alpha_2 Y_p(x). \tag{50}$$

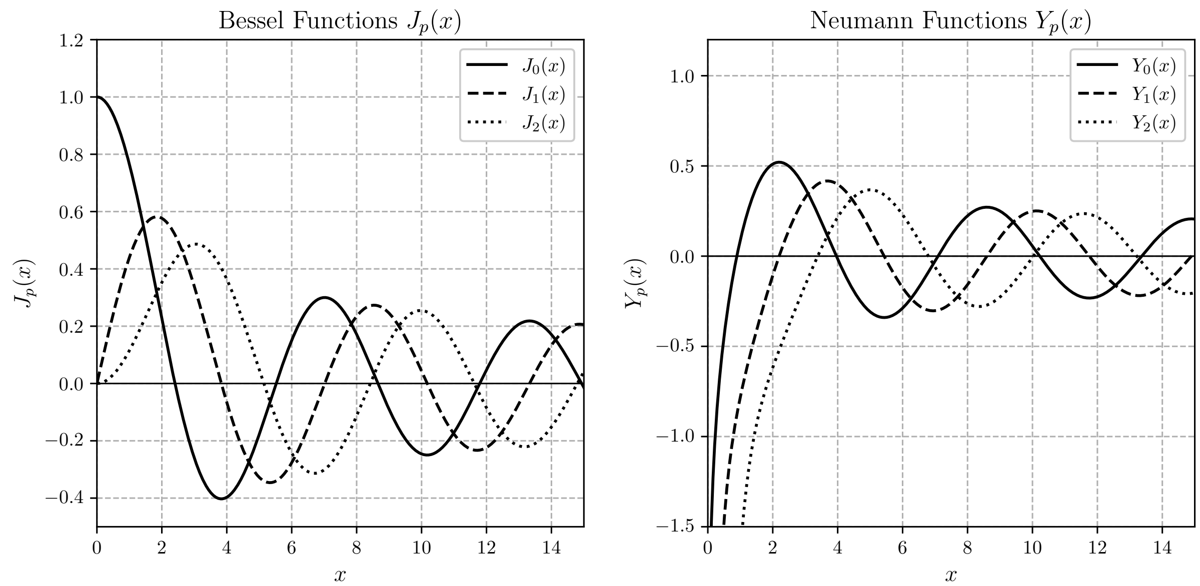

Figure 1: Plots of the Bessel functions of the first kind $J_p(x)$ and the Bessel functions of the second kind (Neumann functions) $Y_p(x)$ for orders $p = 0, 1, 2$.

5 General solution of the Bessel Equation $\forall \nu$

For physical applications, such as solving partial differential equations (PDEs) for systems with cylindrical geometry, it is convenient to establish a single, universal general solution $\forall \nu$, leaving the rest to the boundary conditions.

Our derivation in Case III solved the problem of the linear dependence between $J_p(x)$ and $J_{-p}(x)$, with $p = 0, 1, 2, \ldots$, by constructing the linearly independent Neumann function, $Y_p(x)$. This allowed us to write the general solution as a linear combination of $J_p(x)$ and $Y_p(x)$ (Eq. (50)).

Thereby, we can also express the solution for any order ν in a similar manner, since $Y_\nu(x)$ is itself in terms of $J_\nu(x)$ and $J_{-\nu}(x)$ when ν is non-integer, and according to Eq. (24) the linear combination of these two functions constitutes the general solution when $\nu \neq 0, 1, 2, \ldots$.

We demonstrate this fact by proposing as a solution:

$$y(x) = A\, J_\nu(x) + B\, Y_\nu(x) \tag{51}$$

and showing that the above expression reduces to Eq. (24) if $\nu \neq 0, 1, 2, \ldots$.

We recall the definition of the Neumann function given by Eq. (27):

$$Y_\nu(x) = \frac{\cos(\nu\pi)\, J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

Inserting it into Eq. (51) results in

$$y(x) = A\, J_\nu(x) + B\, \frac{\cos(\nu\pi)\, J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

Rearranging terms yields

$$y(x) = \left[A + B\frac{\cos(\nu\pi)}{\sin(\nu\pi)}\right] J_\nu(x) + \left[-\frac{B}{\sin(\nu\pi)}\right] J_{-\nu}(x). \tag{52}$$

We define new arbitrary constants as follows:

$$c_1 \equiv A + B\frac{\cos(\nu\pi)}{\sin(\nu\pi)}, \quad c_2 \equiv -\frac{B}{\sin(\nu\pi)}.$$

Substituting c_1 and c_2 into Eq. (52) yields

$$y(x) = c_1\, J_\nu(x) + c_2\, J_{-\nu}(x),$$

which is Eq. (24).

Therefore, we conclude that Eq. (51) is a general solution $\forall \nu$.

A Demonstration of Eq. (39)

In order to demonstrate

$$\lim_{z\to -m} \frac{\psi(z)}{\Gamma(z)} = (-1)^{m+1}\, m!,$$

we make use of the following properties of the Gamma and Digamma functions:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{A.1}$$

which is called the reflection formula, and

$$\psi(1-z) - \psi(z) = \pi\cot(\pi z). \tag{A.2}$$

Thereby, the ratio of the Digamma and Gamma function gives

$$\begin{aligned}
\frac{\psi(z)}{\Gamma(z)} &= \frac{1}{\pi}\sin(\pi z)\,\Gamma(1-z)\,[\psi(1-z) - \pi\cot(\pi z)] \\
&= \frac{1}{\pi}\sin(\pi z)\,\Gamma(1-z)\psi(1-z) - \sin(\pi z)\,\Gamma(1-z)\cot(\pi z) \\
&= \frac{1}{\pi}\sin(\pi z)\,\Gamma(1-z)\psi(1-z) - \cos(\pi z)\,\Gamma(1-z).
\end{aligned}$$

If we take the limit as $z \to -m$, with $m = 0, 1, 2, \ldots$, we obtain

$$\lim_{z\to -m} \frac{\psi(z)}{\Gamma(z)} = \lim_{z\to -m}\left[\frac{1}{\pi}\sin(\pi z)\,\Gamma(1-z)\psi(1-z) - \cos(\pi z)\,\Gamma(1-z)\right] \tag{A.3}$$

$$= \frac{1}{\pi}\lim_{z\to -m}\sin(\pi z)\,\Gamma(1-z)\psi(1-z) - \lim_{z\to -m}\cos(\pi z)\,\Gamma(1-z). \tag{A.4}$$

Since $\Gamma(1-z)\psi(1-z)$ is finite when $z = -m$, then

$$\lim_{z\to -m}\sin(\pi z)\,\Gamma(1-z)\psi(1-z) = 0,$$

because $\sin(\pi m) = 0$.

Therefore, Eq. (A.4) becomes

$$\lim_{z\to -m} \frac{\psi(z)}{\Gamma(z)} = -\lim_{z\to -m}\cos(\pi z)\,\Gamma(1-z).$$

Applying properties of limits and evaluating as a product yields

$$\lim_{z\to -m} \frac{\psi(z)}{\Gamma(z)} = -\cos(\pi m)\,\Gamma(m+1).$$

If $m = 0, 1, 2, \ldots$, then $\Gamma(m+1) = m!$ and $\cos(\pi m) = (-1)^m$.

$$\therefore \quad \lim_{z\to -m} \frac{\psi(z)}{\Gamma(z)} = (-1)^{m+1}\, m!.$$

Bibliography

G. B. Arfken and H. J. Weber. *Mathematical Methods for Physicists.*

N. N. Lebedev. *Special Functions and Their Applications.*

Version History

1.1 (July 2026): Minor wording and typographical improvements.
1.0 (June 2026): Original release.