# Motion of a Particle Constrained to a Rotating Plane

**Diego Valenzuela Castro**

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**Abstract**

This note presents a derivation of the trajectory of a particle constrained to slide along a smooth rotating plane. We approach the problem using Lagrangian Mechanics, determining the Lagrangian and the equation of motion via Euler-Lagrange. Finally, we solve the resulting ODE by applying the method of undetermined coefficients, obtaining the time evolution of the position of the particle.

## 1 Description of the Problem

A particle of mass $m$ rests within a smooth plane. While the particle is constrained to it, the plane starts to rotate at a constant angular speed from a horizontal position, causing the particle to move down. The initial position of the particle is at a distance $d$ from the axis of rotation of the plane. The objective is to determine the trajectory of the particle as a function of time.

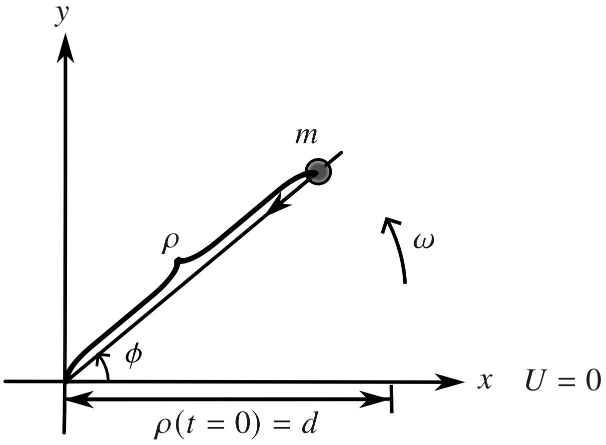

Figure 1: *Particle constrained to a rotating plane*. Due to the nature of the system, polar coordinates are optimal for representing the position of the particle, as explained in Section 2.

## 2 Derivation of the Lagrangian Function

We apply Lagrangian Mechanics to determine the motion of the particle, hence avoiding fictitious forces characteristic of non-inertial systems. Following this method involves the derivation of its kinetic and potential energies, $T$ and $U$, respectively. The calculation of both requires the position vector, and the nature of the degrees of freedom of the system yields the convenient coordinates to express it.

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**ORCID:** https://orcid.org/0009-0006-9386-0692
**Contact:** `diegovc.phys@gmail.com`
**Project:** *Domina la Física* — https://dominalafisica.com

Therefore, it is convenient to choose polar coordinates to describe the motion of the particle (see Fig. 1), since it naturally depends on the distance to the origin and the angular position of the plane as it rotates.

$$\mathbf{r} = x\,\hat{\imath} + y\,\hat{\jmath}$$

$$\mathbf{r} = \rho\cos\phi\,\hat{\imath} + \rho\sin\phi\,\hat{\jmath}\,. \tag{1}$$

Recalling that the gravitational potential energy is given by $U = mgy$, where $y$ is the perpendicular distance from the zero potential reference ($x$ axis), we obtain:

$$U = mg\rho\sin\phi\,. \tag{2}$$

To determine the kinetic energy, we must derive the square modulus of the velocity; thus we take the derivative with respect to time of the position vector.

$$\dot{\mathbf{r}} = \left(\dot\rho\cos\phi - \rho\dot\phi\sin\phi\right)\hat{\imath} + \left(\dot\rho\sin\phi + \rho\dot\phi\cos\phi\right)\hat{\jmath}\,.$$

Note that because $\rho$ and $\phi$ are functions of time, it is necessary to apply the product and chain rules. Taking the square modulus gives:

$$|\dot{\mathbf{r}}|^2 = \left(\dot\rho\cos\phi - \rho\dot\phi\sin\phi\right)^2 + \left(\dot\rho\sin\phi + \rho\dot\phi\cos\phi\right)^2$$

$$|\dot{\mathbf{r}}|^2 = \dot\rho^2\cos^2\phi - 2\rho\dot\rho\dot\phi\cos\phi\sin\phi + \rho^2\dot\phi^2\sin^2\phi + \dot\rho^2\sin^2\phi + 2\rho\dot\rho\dot\phi\cos\phi\sin\phi + \rho^2\dot\phi^2\cos^2\phi\,.$$

We cancel the $2\rho\dot\rho\dot\phi\cos\phi\sin\phi$ terms, which results in:

$$|\dot{\mathbf{r}}|^2 = \dot\rho^2\cos^2\phi + \rho^2\dot\phi^2\sin^2\phi + \dot\rho^2\sin^2\phi + \rho^2\dot\phi^2\cos^2\phi\,.$$

We group the terms by common factor as follows:

$$|\dot{\mathbf{r}}|^2 = \dot\rho^2\underbrace{\left(\cos^2\phi + \sin^2\phi\right)}_{1} + \rho^2\dot\phi^2\underbrace{\left(\cos^2\phi + \sin^2\phi\right)}_{1}\,.$$

Recalling that the kinetic energy is given by $T = \frac{1}{2}m|\dot{\mathbf{r}}|^2$, this implies:

$$T = \frac{1}{2}m\left(\dot\rho^2 + \rho^2\dot\phi^2\right)\,. \tag{3}$$

We introduce the Lagrangian for a conservative system:

$$\mathcal{L} = T - U\,. \tag{4}$$

Substituting the results of Eqs. (2) and (3) into Eq. (4), we obtain the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\left(\dot\rho^2 + \rho^2\dot\phi^2\right) - mg\rho\sin\phi\,. \tag{5}$$

## 3 The Radial Equation of Motion and the Angular Degree of Freedom

We invoke the Euler-Lagrange equations to analyze the dynamics of the system.

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial\mathcal{L}}{\partial\dot\rho}\right) = \dfrac{\partial\mathcal{L}}{\partial\rho} \\ \dfrac{d}{dt}\left(\dfrac{\partial\mathcal{L}}{\partial\dot\phi}\right) = \dfrac{\partial\mathcal{L}}{\partial\phi} \end{cases},$$

where we have one equation for each degree of freedom of the system.

However, note that it is valid to treat the radial degree of freedom as the only one, since we know that the plane rotates at constant angular speed. Therefore, it is possible to determine $\phi$ as an explicit function of time from:

$$\dot{\phi} = \omega \,. \tag{6}$$

Integrating both sides with respect to time yields:

$$\int \dot{\phi}\, dt = \int \omega\, dt$$

$$\phi(t) = \omega t + c \,. \tag{7}$$

As established, the plane starts at a horizontal position, i.e. $\phi(t = 0) = 0$. Applying this initial condition to Eq. (7) results in:

$$\phi(t = 0) = c = 0$$

$$\therefore\ \phi(t) = \omega t \,. \tag{8}$$

This means that, due to the constraint, the angular position of the particle is the same as that of the plane.

We substitute Eqs. (6) and (8) into Eq. (5) to obtain:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{\rho}^2 + \omega^2 \rho^2\right) - m g \rho \sin(\omega t) \,. \tag{9}$$

Therefore, the problem reduces to the radial Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\rho}}\right) = \frac{\partial \mathcal{L}}{\partial \rho} \,. \tag{10}$$

From the Lagrangian of Eq. (9), we evaluate each term of Eq. (10) separately, as shown:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{\rho}} &= \frac{\partial}{\partial \dot{\rho}}\left[\frac{1}{2} m \left(\dot{\rho}^2 + \omega^2 \rho^2\right) - m g \rho \sin(\omega t)\right] \\ &= m\dot{\rho}\end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\rho}}\right) = m\ddot{\rho} \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial \rho} = \frac{\partial}{\partial \rho}\left[\frac{1}{2} m \left(\dot{\rho}^2 + \omega^2 \rho^2\right) - m g \rho \sin(\omega t)\right]$$

$$\frac{\partial \mathcal{L}}{\partial \rho} = m\left[\omega^2 \rho - g \sin(\omega t)\right] \,. \tag{12}$$

Substituting Eqs. (11) and (12) in Eq. (10) yields:

$$m\ddot{\rho} = m\left[\omega^2 \rho - g \sin(\omega t)\right] \,.$$

By canceling the mass $m$ and reordering terms, we obtain the following equation of motion:

$$\ddot{\rho} - \omega^2 \rho = -g \sin(\omega t) \,. \tag{13}$$

# 4 Solution to the Equation of Motion

Eq. (13) is a second-order non-homogeneous linear ODE, which can be solved by the method of undetermined coefficients. In this method, the solution is given by:

$$\rho = \rho_h + \rho_p \,. \tag{14}$$

Here, the solution $\rho$ is the sum of the solution to the homogeneous equation and a particular solution proposed according to the method of undetermined coefficients, $\rho_h$ and $\rho_p$, respectively.

From Eq. (13) we obtain the following homogeneous equation:

$$\ddot{\rho_h} - \omega^2 \rho_h = 0 \,. \tag{15}$$

Eq. (15) is a standard ODE, whose solution is in terms of real exponentials or hyperbolic sines and cosines. Due to the nature of the initial conditions, it is convenient to choose the hyperbolic solution:

$$\rho_h = c_1 \cosh(\omega t) + c_2 \sinh(\omega t) \,. \tag{16}$$

Given that the non-homogeneous term is proportional to $\sin(\omega t)$, we propose:

$$\rho_p = A\cos(\omega t) + B\sin(\omega t) \,. \tag{17}$$

We substitute the particular solution $\rho_p$ in Eq. (13) to determine the values of $A$ and $B$:

$$\frac{d^2}{dt^2}\left[A\cos(\omega t) + B\sin(\omega t)\right] - \omega^2\left[A\cos(\omega t) + B\sin(\omega t)\right] = -g\sin(\omega t)$$

$$-\omega^2\left[A\cos(\omega t) + B\sin(\omega t)\right] - \omega^2\left[A\cos(\omega t) + B\sin(\omega t)\right] + g\sin(\omega t) = 0 \,.$$

Multiplying by -1 and grouping the terms with sine and cosine results in:

$$2\omega^2 A\cos(\omega t) + (2\omega^2 B - g)\sin(\omega t) = 0 \,.$$

Due to the linear independence of $\cos(\omega t)$ and $\sin(\omega t)$, their coefficients must be zero, yielding the following values for A and B:

$$\begin{cases} A = 0 \\ B = \dfrac{g}{2\omega^2} \,. \end{cases}$$

Therefore, substituting the values of $A$ and $B$ in Eq. (17) gives us the particular solution:

$$\rho_p = \frac{g}{2\omega^2}\sin(\omega t) \,. \tag{18}$$

Having determined the homogeneous solution $\rho_h$ and the particular solution $\rho_p$ of Eqs. (16) and (18), respectively, we substitute into the solution proposed in Eq. (14), which results in:

$$\rho(t) = c_1\cosh(\omega t) + c_2\sinh(\omega t) + \frac{g}{2\omega^2}\sin(\omega t) \,. \tag{19}$$

We apply the following initial conditions to $\rho(t)$:

$$\begin{cases} \rho(t=0) = d \\ \dot{\rho}(t=0) = 0 \,. \end{cases}$$

Note that $\rho(t = 0) = d$ and $\dot{\rho}(t = 0) = 0$, because the particle starts from rest at a distance $d$ from the origin.

$$\rho(t=0) = c_1 \underbrace{\cosh 0}_{1} + c_2 \underbrace{\sinh 0}_{0} + \frac{g}{2\omega^2} \underbrace{\sin 0}_{0} = d$$

$$\therefore \; c_1 = d \,.$$

To apply the other initial condition, we take the time derivative of $\rho(t)$ in Eq. (19).

$$\dot{\rho}(t) = c_1\omega \sinh(\omega t) + c_2\omega \cosh(\omega t) + \frac{g}{2\omega} \cos(\omega t)$$

$$\dot{\rho}(t=0) = c_1\omega \underbrace{\sinh 0}_{0} + c_2\omega \underbrace{\cosh 0}_{1} + \frac{g}{2\omega} \underbrace{\cos 0}_{1} = 0$$

$$c_2\omega + \frac{g}{2\omega} = 0$$

$$\therefore \; c_2 = -\frac{g}{2\omega^2} \,.$$

We substitute the calculated values of $c_1$ and $c_2$ into Eq. (19).

$$\rho(t) = d\cosh(\omega t) - \frac{g}{2\omega^2} \sinh(\omega t) + \frac{g}{2\omega^2} \sin(\omega t) \,. \tag{20}$$

By simplifying Eq. (20), we determine the following expression for the radial motion.

$$\rho(t) = d\cosh(\omega t) + \frac{g}{2\omega^2} \left[\sin(\omega t) - \sinh(\omega t)\right] \,. \tag{21}$$

## 5 Trajectory of the Particle

Combining the results of Eqs. (8) and (21), we obtain the angular and radial behavior of the particle.

$$\begin{cases} \phi(t) = \omega t \\ \rho(t) = d\cosh(\omega t) + \dfrac{g}{2\omega^2} \left[\sin(\omega t) - \sinh(\omega t)\right] \,. \end{cases}$$

We recall the position vector introduced in Eq. (1):

$$\mathbf{r} = \rho \cos\phi \,\hat{\imath} + \rho \sin\phi \,\hat{\jmath} \,.$$

Finally, we have determined the trajectory of the particle constrained to the rotating plane.

$$\mathbf{r}(t) = \left\{ d\cosh(\omega t) + \frac{g}{2\omega^2} \left[\sin(\omega t) - \sinh(\omega t)\right] \right\} \left[\cos(\omega t)\,\hat{\imath} + \sin(\omega t)\,\hat{\jmath}\right] \,. \tag{22}$$

## Suggested Reading

S. T. Thornton and J. B. Marion. *Classical Dynamics of Particles and Systems.*

L. D. Landau and E. M. Lifshitz. *Mechanics. Course of Theoretical Physics, Volume 1.*